# Recitation #9: Window-based FIR Filter Design

**Objective & Outline**

- Problems $1-4$: recitation problems
- Problem 5: self-assessment problem

The problems start on the following page.

**Problem 1** (FIR Filter Design)**.** Suppose that we were designing a low-pass filter using the rectangular window method. The desired specifications for this filter are as follows:

- Passband cutoff frequency: $\omega_p = 0.3\pi$
- Stopband cutoff frequency: $\omega_s = 0.5\pi$

(a) Determine the cutoff frequency ($\omega_c$) of this filter.

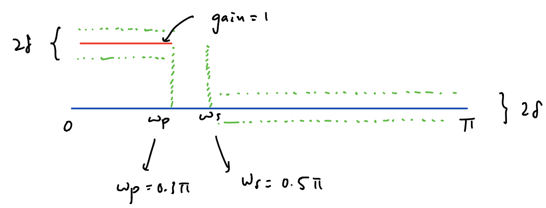
(b) Determine the minimum length (N) of this filter.

(c) Determine the mainlobe width ($\Delta_{ML}$) of this filter.

(d) What is the *designed* impulse response of this filter?

## Problem 1

Let's first consider a rough sketch of this low-pass filter:




(a) The cutoff frequency is simply the average of the passband and stopband frequencies:

$$\omega_c = \frac{\omega_p + \omega_s}{2}$$

$$= \frac{0.3\pi + 0.5\pi}{2} = 0.4\pi.$$

(b) The transition width $(\Delta\omega)$ is the difference between the passband and stopband cutoff frequencies:

$$\Delta\omega = \omega_s - \omega_p$$

$$= 0.5\pi - 0.3\pi = 0.2\pi.$$

Recall that the transition width for a rectangular window is given by

$$\Delta\omega = \frac{0.92\pi}{M}.$$

Thus,

$$M = \frac{0.92\pi}{\Delta\omega}$$

$$= \frac{0.92\pi}{0.2\pi} = \lceil 4.6 \rceil = 5.$$

The minimum length of this filter is given by

$$N = 2M+1 = 2(5)+1 = 11.$$

(c) The mainlobe width $(\Delta_{ML})$ of a rectangular window is given by the equation

$$\Delta_{ML} = \frac{4\pi}{2M+1},$$

where $N = 2M+1$ is the length of the filter. Thus,

$$\Delta_{ML} = \frac{4\pi}{11} = 0.36\pi.$$

(d) Recall that the ideal impulse response of a LPF is given by

$$h_d[n] = \frac{\sin(\omega_c n)}{\pi n},$$

where $\omega_c$ is the cutoff frequency. With $\omega_c = 0.4\pi$,

$$h_d[n] = \frac{\sin(0.4\pi n)}{\pi n}.$$

Now, we need to shift this filter to make it causal:

$$\tilde{h}[n] = h_d[n-M] = \frac{\sin(0.4\pi(n-5))}{\pi(n-5)}.$$

Using the rectangular window, $h[n] = \tilde{h}[n]\, w[n]$:

$$h[n] = \begin{cases} \frac{\sin(0.4\pi(n-5))}{\pi(n-5)}, & 0 \le n \le 10 \\ 0, & \text{otherwise}. \end{cases}$$

$\blacksquare$

**Problem 2** (FIR Filter Design)**.** Suppose we were designing a bandstop FIR filter using the rectangular window method. The design specifications for our filter are given as follows:

- Passband gain: $G = 2$
- Passband frequencies: $\omega_{p_1} = 0.56\pi$ and $\omega_{p_2} = 0.83\pi$
- Stopband frequencies: $\omega_{s_1} = 0.64\pi$ and $\omega_{s_2} = 0.74\pi$

(a) What is the smallest length of the filter (equivalently, the rectangular window) that will result in an FIR filter with the desired specifications?

(b) What will be the maximum (absolute) values of the ripples in passband ($\delta_p$) and stopband ($\delta_s$)?

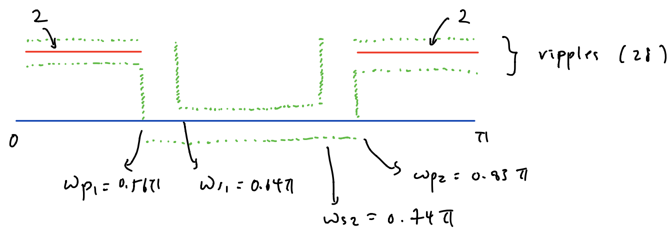
(c) Provide a closed-form expression for the impulse response of the *designed* FIR filter.

(d) How many poles and zeros will this filter have?

(e) What is the *type* of this FIR filter?

Problem 2:

Recall the figure for a bandstop filter with DT frequencies:




(a) To find the minimum filter length, we first need to compute the transition widths:

$$\Delta\omega_1 = \omega_{s_1} - \omega_{p_1} = 0.64\pi - 0.56\pi = 0.08\pi$$

$$\Delta\omega_2 = \omega_{s_2} - \omega_{p_2} = 0.83\pi - 0.74\pi = 0.09\pi.$$

When we have two transition widths, we want the smallest of the two to be more strict:

$$\Delta\omega = \min\{0.08\pi,\ 0.09\pi\} = 0.08\pi.$$

Since we are using the rectangular window method:

$$M = \frac{0.92\pi}{\Delta\omega} = \lceil 11.5 \rceil = 12.$$

Thus, the length is

$$N = 2M + 1 = 2(12) + 1 = 25.$$

(b) For a rectangular window, the maximum relative ripples in magnitude is

$$\alpha_s = 20.9\ dB.$$

Solving for $\delta$, we have

$$20.9 = -20\log_{10}(\delta/2)$$

$$\Rightarrow -1.045 = \log_{10}(\delta/2)$$

$$\Rightarrow 10^{(-1.045)} \cdot 2 \Rightarrow \delta = 0.1802.$$

(c) The two cutoff frequencies are given by

$$\omega_{c1} = \frac{\omega_{s1} + \omega_{p1}}{2} = 0.6\pi$$

$$\omega_{c2} = \frac{\omega_{s2} + \omega_{p2}}{2} = 0.785\pi.$$

Recall that $H(e^{j\omega})$ for an FIR bandstop filter is given by

$$H(e^{j\omega}) = 1 - H_{BP}^{\omega_{c1}, \omega_{c2}}(e^{j\omega}).$$

In discrete-time, the impulse response is

$$h_d[n] = \delta[n] - \left[ \frac{\sin(\omega_{c2} n)}{\pi n} - \frac{\sin(\omega_{c1} n)}{\pi n} \right].$$

$$= \begin{cases} 1 - \frac{(\omega_{c2} - \omega_{c1})}{\pi}, & \text{for } n = 0 \\ \frac{\sin(\omega_{c1} n)}{\pi n} - \frac{\sin(\omega_{c2} n)}{\pi n}, & \text{for } n \neq 0. \end{cases}$$

Then, we need to shift by M to make the filter causal:

$$\tilde{h}[n] = h_d[n - M]$$

$$= \begin{cases} 1 - \frac{(\omega_{c2} - \omega_{c1})}{\pi}, & \text{for } n = 12 \\ \frac{\sin(\omega_{c1}(n-12))}{\pi(n-12)} - \frac{\sin(\omega_{c2}(n-12))}{\pi(n-12)}, & \text{for } n \neq 12. \end{cases}$$

Using a rectangular window,

$$h[n] = \begin{cases} 0.815, & \text{for } n = 12 \\ \frac{\sin(0.6\pi(n-12))}{\pi(n-12)} - \frac{\sin(0.785\pi(n-12))}{\pi(n-12)}, & \text{for } n \in [0,12) \cup (12, 24]. \\ 0, & \text{otherwise}. \end{cases}$$

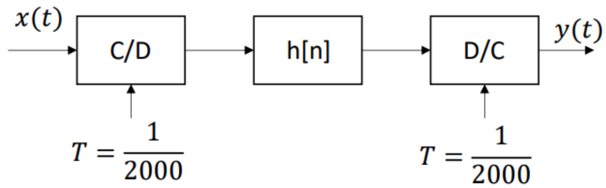
(d) We need to have

$$\# \text{ of poles} = \# \text{ of zeroes} = N - 1$$
$$= 24.$$

(e) Since $h[n]$ is odd length and symmetric, this is a

Type 1 filter. ∎

**Problem 3** (FIR Filter Design). Consider the following block diagram of a DSP system:




It is desired that the equivalent system act as a low-pass filter with a cutoff frequency of 500 Hz. Suppose $x(t)$ is bandlimited to 1kHz. Design a rectangular window-based FIR filter with a length of 7 to achieve the desired system.

(a) Determine the *desired* impulse response, $h_d[n]$.

(b) Determine the *designed* impulse response, $h[n]$.

(c) What *type* of filter is $h[n]$?

Problem 3:

(a) We first need to determine the cutoff frequency:

$$\begin{aligned} \omega_c &= 2\pi f_c T \\ &= 2\pi (500) \left(\frac{1}{2000}\right) \\ &= \pi/2 . \end{aligned}$$

Since we want a low-pass filter,

$$h_d[n] = \frac{\sin\left(\frac{\pi}{2} n\right)}{\pi n} .$$

(b) We need to find 'M' given our length:

$$N = 2M + 1$$

$$\Rightarrow \quad M = \frac{N-1}{2} = 3 .$$

The shifted impulse response is

$$\begin{aligned} \tilde{h}[n] &= h_d[n-M] \\ &= \frac{\sin\left(\frac{\pi}{2}(n-3)\right)}{\pi(n-3)} \end{aligned}$$

Using a rectangular window,

$$h[n] = \begin{cases} \tilde{h}[n] , & n = 0, 1, \ldots, 6 \\ 0 , & \text{otherwise} \end{cases}$$

$$= \{ -0.11, \ 0.32, \ 0.5, \ 0.32, \ 0, \ -0.11 \} .$$

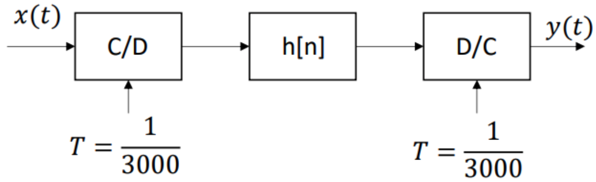
(c) Since $h[n]$ is odd length and symmetric, $h[n]$ is a

Type 1 filter.

Note that the system acts as an LTI system as there was no aliasing:

$$2000 \text{ kHz} \geq 2000 \text{ kHz} .$$

**Problem 4** (FIR Filter Design)**.** Consider the following block diagram of a DSP system:




Suppose the input $x(t)$ was bandlimited to 1kHz. It is desired that the equivalent system act as a low-pass filter with the following specifications:

- Passband gain: $G = 5$
- Passband cutoff: $f_p = 400\text{Hz}$
- Stopband cutoff: $f_s = 600\text{Hz}$
- Passband and stopband ripples: $\delta = 0.1$

(a) What are the specifications of this digital filter in discrete-time?

(b) Express the size of the ripples in dB.

(c) What is the ideal (or desired) impulse response of this digital filter?

Problem 4:

(a) In discrete time, the system is still a low-pass filter, but the cutoff frequencies are now

$$\omega_p = 2\pi f_p T$$
$$= 2\pi (400)\left(\frac{1}{3000}\right) = \frac{4\pi}{15}.$$

$$\omega_s = 2\pi f_s T$$
$$= 2\pi (600)\left(\frac{1}{3000}\right) = \frac{2\pi}{5}.$$

The transition width is

$$\Delta\omega = \omega_p - \omega_s = \frac{2\pi}{15}.$$

The gain and ripple values stay the same:

$$\text{Gain} = 5$$
$$\text{ripples } (\delta) = 0.1.$$

(b) Using the equation:

$$\alpha_s = -20 \log_{10}\left(\frac{0.1}{5}\right)$$

$$\approx -34 \text{ dB gain.}$$

(c) The cutoff frequency is given by

$$\omega_c = \frac{\omega_p + \omega_s}{2} = \frac{\pi}{3}.$$

Thus,

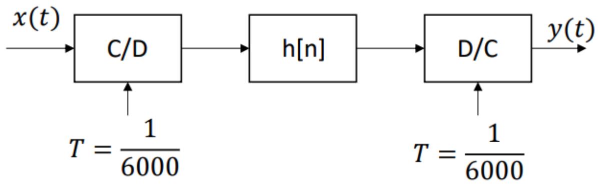
$$h_d[n] = 5\,\frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n}.$$

Note that the system acts as an LTI system as there was no aliasing:

$$3000 \text{ kHz} \geq 2000 \text{ kHz}.$$

**Problem 5** (Self-assessment)**.** As usual, try to work on these problems together in break-out rooms.

Consider the following block diagram of a DSP system:




It is desired that the equivalent system act as a high-pass filter with a cutoff frequency of 1kHz and gain 2. Suppose $x(t)$ is bandlimited to 3kHz. Design a rectangular window-based FIR filter with a length of 7 to achieve the desired system.

(a) What are the specifications of this digital filter in discrete-time?

(b) Determine the *desired* impulse response, $h_d[n]$.

(c) Determine the *designed* impulse response, $h[n]$.

(d) What *type* of filter is $h[n]$?

## Problem 5:

(a) Similar to Problem 4:

$$\begin{aligned} \omega_c &= 2\pi f_c T \\ &= 2\pi (1000) \left(\frac{1}{6000}\right) \\ &= \pi/3 \end{aligned}$$

$$\text{Gain} = 2.$$

(b) The desired impulse response is given by

$$\begin{aligned} h_d[n] &= 2\delta[n] - \frac{2\sin(\frac{\pi}{3}n)}{\pi n} \\ &= \begin{cases} 4/3, & n = 0 \\ -\frac{2\sin(\pi/3 n)}{\pi n}, & n \neq 0. \end{cases} \end{aligned}$$

(c) Shifting and windowing $h_d[n]$, we get

$$\begin{aligned} \tilde{h}[n] &= h_d[n-M] \\ &= 2\delta[n-3] - \frac{2\sin(\frac{\pi}{3}(n-3))}{\pi(n-3)} \\ &= \begin{cases} 4/3, & n = 3 \\ -\frac{2\sin(\pi/3 n)}{\pi n}, & n \neq 3. \end{cases} \end{aligned}$$

Thus,

$$h[n] = \begin{cases} 4/3, & n = 3 \\ \frac{-2\sin(\pi/3 n)}{\pi n}, & n \in [0,3) \cup (3,6] \\ 0, & \text{otherwise}. \end{cases}$$

Note that you could have left your answer in terms of $\delta[n]$:

(d) Since $h[n]$ is of odd length and symmetric, this is a

Type I Filter.

Note that the system acts as an LTI system as there was no aliasing.

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